LU Decomposition

Formula :

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \quad and \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Example : $\quad Solve\ Ax = B, where\ A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}\ and\ B = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

Solution : $\quad Let, \quad LU = A$

$$So, \quad \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \times \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + l_{32}U_{23} + U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

$U_{11} = 1$

$U_{12} = 1$

$U_{13} = 1$

$L_{21}U_{11} = 4 \quad or, L_{21} \times 1 = 4 \quad or, L_{21} = 4$

$L_{21}U_{12} + U_{22} = 3 \quad or, (4 \times 1) + U_{22} = 3 \quad or, U_{22} = -1$

$L_{21}U_{13} + U_{23} = -1 \quad or, (4 \times 1) + U_{23} = -1 \quad or, U_{23} = -5$

$L_{31}U_{11} = 3 \quad or, L_{31} \times 1 = 3 \quad or, L_{31} = 3$

$L_{31}U_{12} + L_{32}U_{22} = 5 \quad or, (3 \times 1) + \big(L_{32} \times (-1)\big) = 5 \quad or, L_{32} = -2$

$L_{31}U_{13} + l_{32}U_{23} + U_{33} = 3 \quad or, (3 \times 1) + \big((-2) \times (-5)\big) + U_{33} = 3 \quad or, U_{33} = -10$

$$So, \quad Ax = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}} \times \underset{U}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}} \times \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}}$$

$Set\ y = Ux\ so\ that\ Ax = Ly = B$

$$\Rightarrow \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}} \times \underset{y}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ 4y_1 + y_2 \\ 3y_1 - 2y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$y_1 = 1$

$4y_1 + y_2 = 6 \qquad or, (4 \times 1) + y_2 = 6 \qquad or, y_2 = 2$

$3y_1 - 2y_2 + y_3 = 4 \qquad or, (3 \times 1) - (2 \times 2) + y_3 = 4 \qquad or, y_3 = 5$

$$So, \quad \overset{U}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}} \times \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{y}{\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} x_1 + x_2 + x_3 \\ -x_2 - 5x_3 \\ -10x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$-10x_3 = 5 \qquad or, x_3 = -\frac{1}{2}$

$-x_2 - 5x_3 = 2 \qquad or, -x_2 - \left(5 \times \left(-\frac{1}{2}\right)\right) = 2 \qquad or, x_2 = \frac{1}{2}$

$x_1 + x_2 + x_3 = 1 \qquad or, x_1 + \frac{1}{2} - \frac{1}{2} = 1 \qquad or, x_1 = 1$

$$So, \quad x = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$Therefore, \quad Ax = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} \ (proved)$$