Gauss Elimination

This approach is designed to solve general set of n equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots\ldots + a_{3n}x_n = b_3$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots\ldots + a_{nn}x_n = b_n$$

Represent in matrix form :

$$\begin{vmatrix} a_{11}x_1 & a_{12}x_2 & a_{13}x_3 & \ldots\ldots\ldots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & a_{23}x_3 & \ldots\ldots\ldots & a_{2n}x_n \\ a_{31}x_1 & a_{32}x_2 & a_{33}x_3 & \ldots\ldots\ldots & a_{3n}x_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1}x_1 & a_{n2}x_2 & a_{n3}x_3 & \ldots\ldots\ldots & a_{nn}x_n \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ b_n \end{vmatrix}$$

Consider 3 *equation to understand gauss elemination method*

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Represent this 3 *equation in matrix form* :

$$\begin{vmatrix} a_{11}x_1 & a_{12}x_2 & a_{13}x_3 \\ a_{21}x_1 & a_{22}x_2 & a_{23}x_3 \\ a_{31}x_1 & a_{32}x_2 & a_{33}x_3 \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix}$$

We need to represent above matrix in the form of :

$$\begin{vmatrix} a_{11}x_1 & a_{12}x_2 & a_{13}x_3 \\ 0 & a_{22}x_2 & a_{23}x_3 \\ 0 & 0 & a_{33}x_3 \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix}$$

Consider 4 *equation to understand gauss elemination method*

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3$$
$$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4$$

Represent this 3 *equation in matrix form* :

$$\begin{vmatrix} a_{11}x_1 & a_{12}x_2 & a_{13}x_3 & a_{14}x_4 \\ a_{21}x_1 & a_{22}x_2 & a_{23}x_3 & a_{24}x_4 \\ a_{31}x_1 & a_{32}x_2 & a_{33}x_3 & a_{34}x_4 \\ a_{41}x_1 & a_{42}x_2 & a_{43}x_3 & a_{44}x_4 \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{vmatrix}$$

We need to represent above matrix in the form of :

$$\begin{vmatrix} a_{11}x_1 & a_{12}x_2 & a_{13}x_3 & a_{14}x_4 \\ 0 & a_{22}x_2 & a_{23}x_3 & a_{24}x_4 \\ 0 & 0 & a_{33}x_3 & a_{34}x_4 \\ 0 & 0 & 0 & a_{44}x_4 \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{vmatrix}$$

Example : *Use Gauss Elimination to solve*

$$10x_1 - 7x_2 = 7$$
$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$
$$5x_1 - x_2 + 5x_3 = 6$$

Solution :

$$\Rightarrow \begin{vmatrix} 10x_1 & -7x_2 & 0 \\ -3x_1 & 2.099x_2 & 6x_3 \\ 5x_1 & -x_2 & 5x_3 \end{vmatrix} = \begin{vmatrix} 7 \\ 3.901 \\ 6 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 10x_1 & -7x_2 & 0 \\ 0 & -0.001x_2 & 6x_3 \\ 5x_1 & -x_2 & 5x_3 \end{vmatrix} = \begin{vmatrix} 7 \\ 6.001 \\ 6 \end{vmatrix} \qquad \left[row2 = row2 - \left(row1 \times \frac{a_{21}}{a_{11}}\right)\right]$$

$$\Rightarrow \begin{vmatrix} 10x_1 & -7x_2 & 0 \\ 0 & -0.001x_2 & 6x_3 \\ 0 & 2.5x_2 & 5x_3 \end{vmatrix} = \begin{vmatrix} 7 \\ 6.001 \\ 2.5 \end{vmatrix} \qquad \left[row3 = row3 - \left(row1 \times \frac{a_{31}}{a_{11}}\right)\right]$$

$$\Rightarrow \begin{vmatrix} 10x_1 & -7x_2 & 0 \\ 0 & -0.001x_2 & 6x_3 \\ 0 & 0 & 15005x_3 \end{vmatrix} = \begin{vmatrix} 7 \\ 6.001 \\ 15005 \end{vmatrix} \qquad \left[row3 = row3 - \left(row2 \times \frac{a_{32}}{a_{22}}\right)\right]$$

From row3 we get $15005x_3 = 15005$

$$\Rightarrow x_3 = \frac{15005}{15005} = 1$$

Put the value of x_3 *into row2*

$$\Rightarrow -0.001x_2 + 6x_3 = 6.001$$
$$\Rightarrow -0.001x_2 + (6 \times 1) = 6.001$$
$$\Rightarrow -0.001x_2 = 6.001 - 6$$
$$\Rightarrow x_2 = \frac{0.001}{-0.001} = -1$$

Put the value of x_2 *into row1*

$$\Rightarrow 10x_1 - 7x_2 = 7$$
$$\Rightarrow 10x_1 - \{7 \times (-1)\} = 7$$
$$\Rightarrow 10x_1 = 7 - 7$$
$$\Rightarrow x_1 = 0$$

The values are $x_1 = 0, \ \ x_2 = -1, \ \ x_3 = 1$